LECTURE 19 e AS A LIMIT AND INVERSE TRIGONOMETRIC FUNCTIONS

Theorem.
$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}.$$

Proof. The proof may be a little top-to-bottom, as in, pretty tricky. We don't start with the limit as stated. Instead, we try to show
$$\ln\left(\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right) = 1.$$
Noting that $\ln(x)$ is a continuous function, we can definitely write the LHS as (by definition of continuity, $\ln(\lim_{x\to a} x) = \lim_{x\to a}\ln(x)$)
$$\begin{aligned}\ln\left(\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right) &= \lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}\\ &= \lim_{x\to 0}\frac{\ln(1+x)}{x}\end{aligned}$$
This starts to look like a derivative, by adding a new term $\ln(1) = 0$,
$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{\ln(1+x)-\ln(1)}{x} = \left(\frac{d}{dx}\ln(x)\right)|_{x=1} = \frac{1}{x}\;|_{x=1} = 1$$
and we are done. $\square$

Example. Evaluate the limit $\lim_{x\to 0}(1+2x)^{\frac{1}{x}}$.

Solution. Certainly, we want to make use of the theorem. How? The idea is to make sure the $2x$ here matches with the exponent as $\frac{1}{2x}$, by doing some tricks. Note that
$$(1+2x)^{\frac{1}{x}} = \left((1+2x)^{\frac{1}{2x}}\right)^2.$$
Thus,
$$\begin{aligned}\lim_{x\to 0}(1+2x)^{\frac{1}{x}} &= \lim_{x\to 0}\left((1+2x)^{\frac{1}{2x}}\right)^2\\ &= \left(\lim_{x\to 0}(1+2x)^{\frac{1}{2x}}\right)^2\\ &\overset{y=2x}{=} \left(\lim_{y\to 0}(1+y)^{\frac{1}{y}}\right)^2\\ &= e^2\end{aligned}$$

Example. Show that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$.

Solution. One can replace n by y. Then,
$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{y\to\infty}\left(1+\frac{x}{y}\right)^y\\ &= \lim_{z\to 0}(1+xz)^{\frac{1}{z}}\\ &= \lim_{z\to 0}\left((1+xz)^{\frac{1}{xz}}\right)^x\\ &= \left(\lim_{z\to 0}(1+xz)^{\frac{1}{xz}}\right)^x\\ &\overset{x\neq 0}{=} \left(\lim_{w\to 0}(1+w)^{\frac{1}{w}}\right)^x\\ &= e^x.\end{aligned}$$

Inverse Trigonometric Functions

When we invert a function, we must be careful of where the original function might have flat slopes. For example, $f(x) = x^2$ is not invertible at the point $x = 0$ because it has a flat tangent line there (flipping it about the origin gives an infinite slope for the flipped point). The same rationale goes for trigonometric functions.

One strategy is to identify part of the domain that makes the function "one-to-one", i.e. given an output, it can only come from one input. For example, $f(x) = x^2$ on $(-\infty, \infty)$ is NOT one-to-one in the sense that, given an output value 16, we have two points ± 4 that can get you this value. However, $f(x) = x^2$ on $(0, \infty)$ IS one-to-one since now for each function value, you can only find one input to map to it.

One-to-one functions are invertible on their respective domains.

Similary, for $f(x) = \sin(x)$, we find that the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ that $\sin(x)$ is one-to-one. Then $\sin(x)$ is invertible, and its inverse is called $\sin^{-1}(x)$. Here we collect a table of results on where the trigonometric function is one-to-one:

Functions	Domain	Range
$\sin(x)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[-1, 1]$
$\cos(x)$	$[0, \pi]$	$[-1, 1]$
$\tan(x)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$(-\infty, \infty)$
$\cot(x)$	$[0, \pi]$	$(-\infty, \infty)$
$\sec(x)$	$[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$	$(-\infty, -1] \cup [1, \infty)$
$\csc(x)$	$[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$	$(-\infty, -1] \cup [1, \infty)$

The corresponding inverse functions thus have the domain and range flipped.

Now, back to derivatives. Since $\sin^{-1}(x)$ has domain $[-1, 1]$, its derivative is only defined there. It naturally shows up in its expression.

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}, \quad |x| < 1.$$

Also, since $\tan^{-1}(x)$ has domain $(-\infty, \infty)$, its derivative is defined everywhere.

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}, \quad x \in (-\infty, \infty).$$

and

$$\frac{d}{dx}\left(\cot^{-1}(x)\right) = -\frac{1}{1+x^2}, \quad x \in (-\infty, \infty).$$

Let's use the same triangle method to find $\frac{d}{dx}\left(\cos^{-1}(x)\right)$ and $\frac{d}{dx}\left(\cot^{-1}(x)\right)$.

The next few are also interesting.

$$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \begin{cases} \frac{1}{x\sqrt{x^2-1}}, & x > 1, \\ -\frac{1}{x\sqrt{x^2-1}}, & x < -1. \end{cases} = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1.$$

and

$$\frac{d}{dx}\left(\csc^{-1}(x)\right) = -\frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1.$$